The Gaussian Copula
A Case Study

Gary Schurman, MBE, CFA

September, 2018

In this white paper we will use a given correlation matrix and then solve for the correlations to use in the Gaussian Copla. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We will define the variable ρ_i to be the correlation of the random return on the i'th asset in the portfolio with a common factor. We want to use the Gaussian Copula to model a portfolio of three correlated assets and therefore need to determine the values of ρ_1, ρ_2 and ρ_3. The table below presents our correlation matrix...

Table 1: Correlation Matrix

	Asset 1	Asset 2	Asset 3
Asset 1	1.00	0.60	0.40
Asset 2	0.60	1.00	0.50
Asset 3	0.40	0.50	1.00

Question: What are the values of ρ_1, ρ_2 and ρ_3?

The Mathematics

Note that the correlation matrix in Table 1 above can be written as follows...

Table 2: Correlation Matrix Composition

	Asset 1	Asset 2	Asset 3
Asset 1	ρ_{11}	ρ_{21}	ρ_{31}
Asset 2	ρ_{12}	ρ_{22}	ρ_{32}
Asset 3	ρ_{13}	ρ_{23}	ρ_{33}

Using Table 2 above note the following equalites...

$$\begin{aligned} &\text{Correlation of Asset 1 and Asset 2: } \rho_{12}=\rho_{21}=0.60\\ &\text{Correlation of Asset 1 and Asset 3: } \rho_{13}=\rho_{31}=0.40\\ &\text{Correlation of Asset 2 and Asset 3: } \rho_{23}=\rho_{32}=0.50 \end{aligned} \tag{1}$$

Given that the correlation of the i'th and j'th asset in the portfolio is equal to the product of the i'th asset's correlation with the common factor and the j'th asset's correlation with the common factor note the following equalites...

$$\begin{aligned} &\text{Correlation of Asset 1 and Asset 2: } \rho_1\times\rho_2=\rho_{12}\\ &\text{Correlation of Asset 1 and Asset 3: } \rho_1\times\rho_3=\rho_{13}\\ &\text{Correlation of Asset 2 and Asset 3: } \rho_2\times\rho_3=\rho_{23} \end{aligned} \tag{2}$$

In Equation (2) above we have three equations that need to be solved simulataneously. We want to use Linear Algebra to accomplish this task but we need three linear equations. We can get these linear equations by taking the natural log of both sides of Equation (2) above as follows...

$$\begin{aligned}
&\text{Log of the Correlation of Asset 1 and Asset 2: } \ln(\rho_1)+\ln(\rho_2)=\ln(\rho_{12})\\
&\text{Log of the Correlation of Asset 1 and Asset 3: } \ln(\rho_1)+\ln(\rho_3)=\ln(\rho_{13})\\
&\text{Log of the Correlation of Asset 2 and Asset 3: } \ln(\rho_2)+\ln(\rho_3)=\ln(\rho_{23})
\end{aligned} \tag{3}$$

We will make the following matrix and vector definitions...

$$\mathbf{A}=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} \text{ ...and... } \vec{\mathbf{b}}=\begin{bmatrix}\ln(\rho_1)\\\ln(\rho_2)\\\ln(\rho_3)\end{bmatrix} \text{ ...and... } \vec{\mathbf{c}}=\begin{bmatrix}\ln(\rho_{12})\\\ln(\rho_{13})\\\ln(\rho_{23})\end{bmatrix} \tag{4}$$

Note: Matrix $\mathbf{A}$ above is a square matrix with number of rows and columns equal to $n!/2!(n-2)!$ where the variable n is the number of assets in the portfolio. Vectors $\mathbf{b}$ and $\mathbf{c}$ above are each column vectors with number of rows equal to n.

Using the definitions in Equation (4) above we can rewrite Equation (3) above as...

$$\mathbf{A}\,\vec{\mathbf{b}}=\vec{\mathbf{c}} \tag{5}$$

Using Equation (5) above not the following equality...

$$\text{if... } \mathbf{A}\,\vec{\mathbf{b}}=\vec{\mathbf{c}} \text{ ...then... } \mathbf{A}^{-1}\,\vec{\mathbf{c}}=\vec{\mathbf{b}} \tag{6}$$

The Point: If we take the product of the inverse of matrix $\mathbf{A}$ (known values) and vector $\vec{\mathbf{c}}$ (known values) then we get vector $\vec{\mathbf{b}}$ (unknown values). Since the elements of vector $\vec{\mathbf{b}}$ are natural logs then to get the actual correlations we take the exponential of each element in that vector as follows...

$$\rho_1=\text{Exp}\Big\{b_1\Big\} \text{ ...and... } \rho_2=\text{Exp}\Big\{b_2\Big\} \text{ ...and... } \rho_3=\text{Exp}\Big\{b_3\Big\} \tag{7}$$

The Answer To Our Hypothetical Problem

Using Equation (4) above and the parameters in Table 1 above we will define the following matrices and vector...

$$\mathbf{A}=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix} \text{ ...such that... } \mathbf{A}^{-1}=\begin{bmatrix}0.50&0.50&-0.50\\0.50&-0.50&0.50\\-0.50&0.50&0.50\end{bmatrix} \text{ ...and... } \vec{\mathbf{c}}=\begin{bmatrix}\ln(0.60)=-0.5108\\\ln(0.40)=-0.9163\\\ln(0.50)=-0.6931\end{bmatrix} \tag{8}$$

Using Equations (6) and (8) above the solution to vector $\mathbf{b}$ is...

$$\vec{\mathbf{b}}=\mathbf{A}^{-1}\,\vec{\mathbf{c}}=\begin{bmatrix}-0.3670\\-0.1438\\-0.5493\end{bmatrix} \tag{9}$$

Using Equations (7) and (9) above the answer to our hypothetical problem is...

$$\rho_1=\text{Exp}\Big\{-0.3670\Big\}=0.6928 \text{ and } \rho_2=\text{Exp}\Big\{-0.1438\Big\}=0.8660 \text{ and } \rho_3=\text{Exp}\Big\{-0.5493\Big\}=0.5774 \tag{10}$$